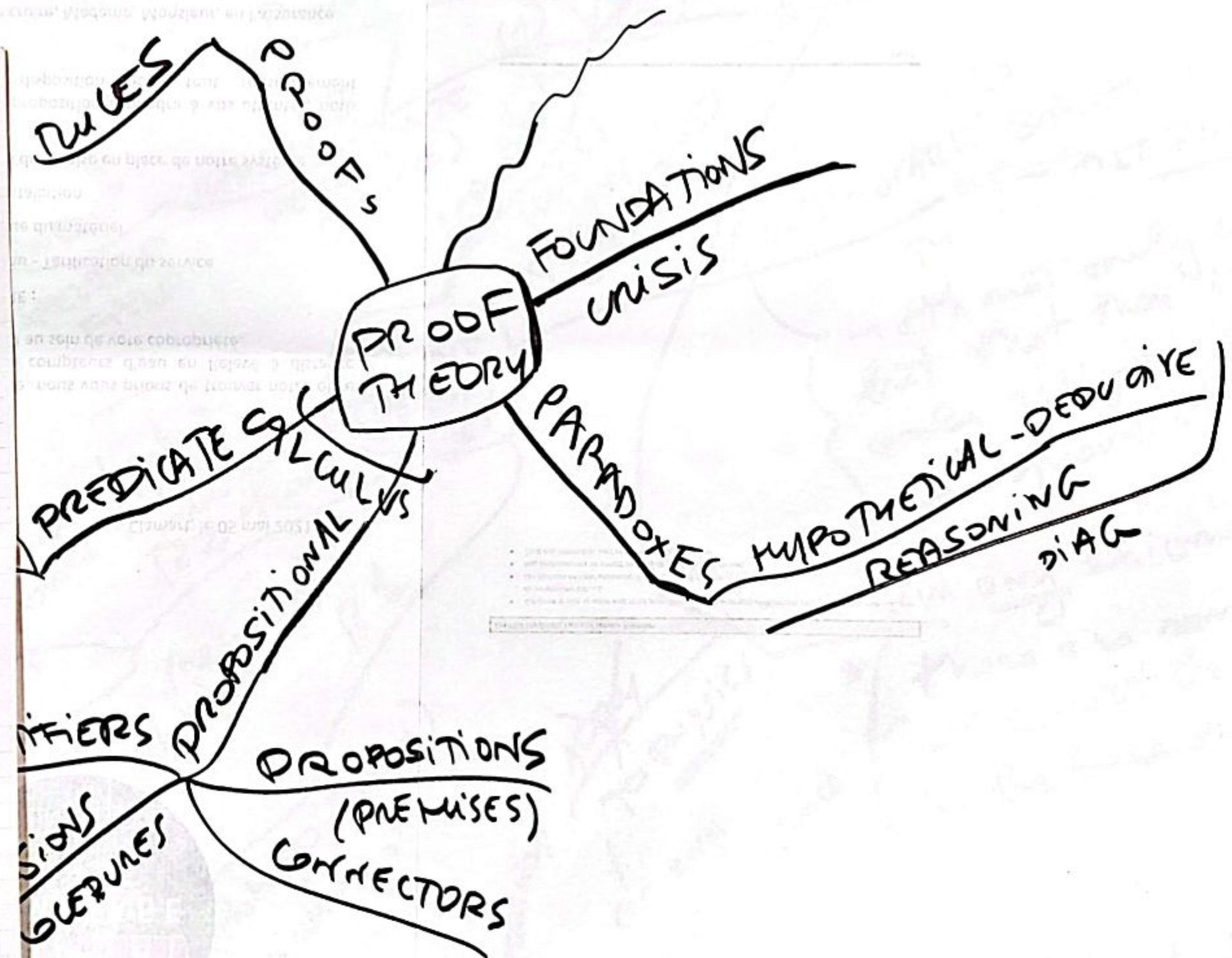
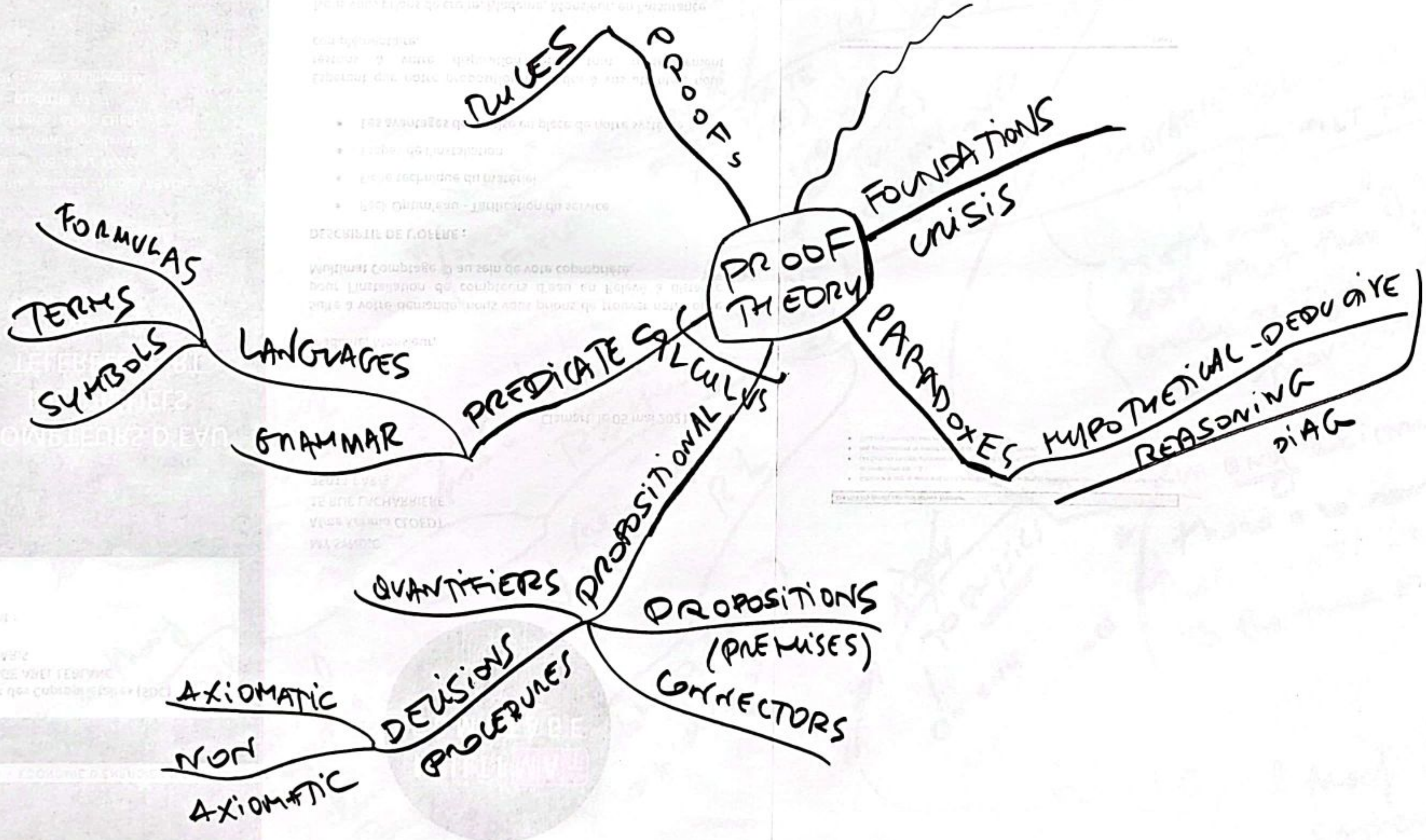
26A



SOZ PROOF THEORY


PROOF THEORY
RULES
PROOFS
FOUNDATIONS
CRISIS
PARADOXES
HYPOTHETICAL-DEDUCTIVE
REASONING
DIAG
PREDICATE CALCULUS
LANGUAGES
FORMULAS
TERMS
SYMBOLS
GRAMMAR
PROPOSITIONAL CALCULUS
QUANTIFIERS
PROPOSITIONS
(PREMISES)
CONNECTORS
DECISIONS PROCEDURES
AXIOMATIC
NON AXIOMATIC

PROOF THEORY 1

(1) INDEP REASON & DEMONSTRATE (PROVE)

* SPECIALIZATION FIELD

PROCESS

INDEP OF LANGUAGE USED *

(2) (3) PREDICATE

(4) — PATH — GODEL'S INCOMPLETENESS THEOREM

MAIN

LOGIC THEORY (LOG SYS)

(5) — (AUTOMATA THEORY)

OF PROP CAL (LOGIC)

5 MAIN OBSERVED

UNPROVABLE STATEMENT ⊃ THEORY ⊃ EXIST FAITH

Math not only a religion, but that then it is the only religion that can prove it is one!

STATE THAT

* in any axiomatic system there are some statements that cannot be determined to be true or false.

~~APPLY~~ TO PHYSICS

or any experience science

~~RELY~~ ON logical proof in axiomatic systems,

it relies on empirical evidence!

but that any proof can be done on a common ~~language~~ starting from a finite number R2 of rules

~~STACK~~

everything is demonstrable

working (reasoning) set R1

IN theoretical computing

including Boolean algebra //

(AUTOMATA theory) (logical system)

PROOF
THEORY
?

paradoxes ↑
⟹ nec[e]ssity
rules of proof
and to verify that
not contradictory
early 20th
"right" to the
in reasoning doubt

origin
axioms
foundations
necessary
to do some given
to progress
on
*

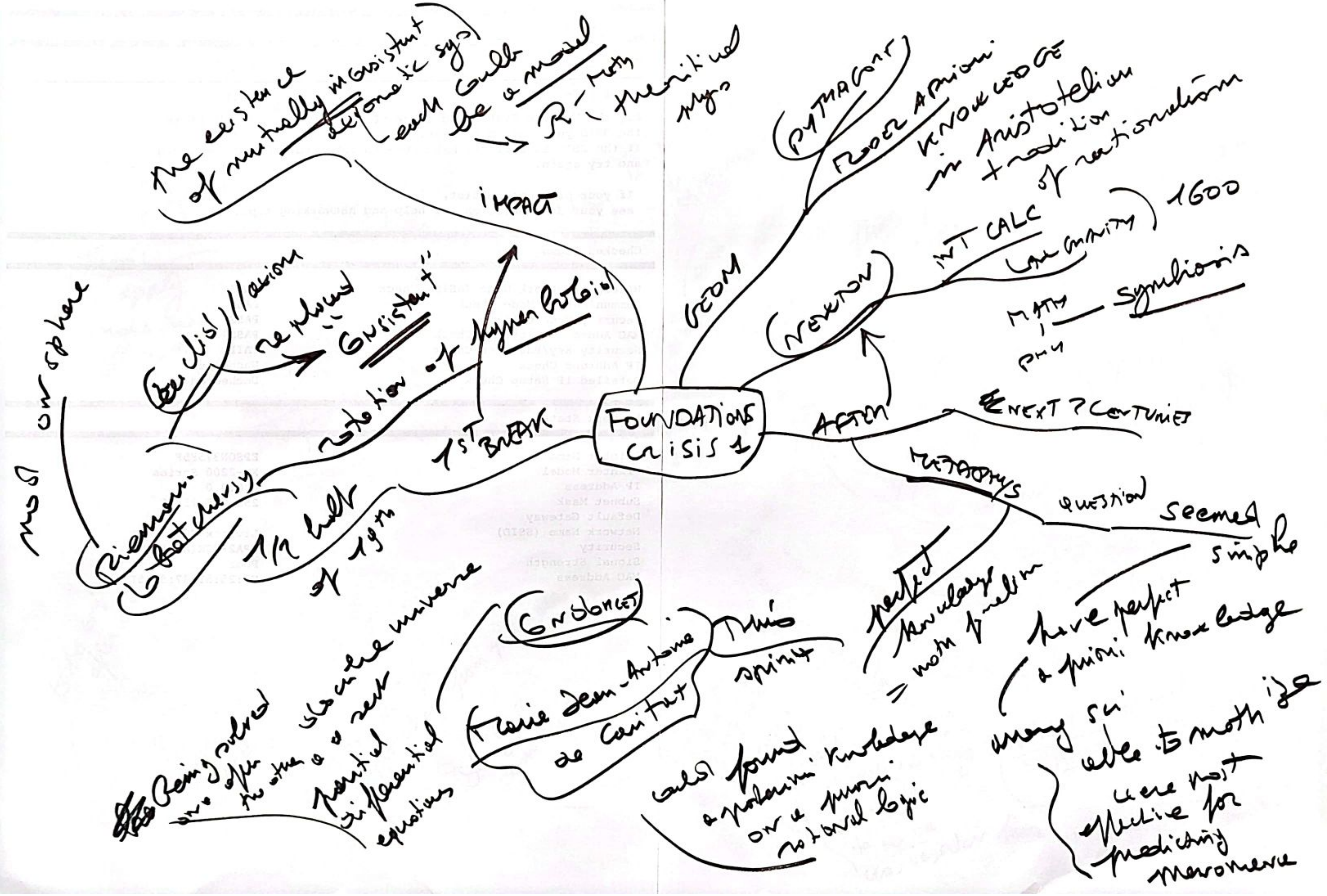
these
Tools ∃
but are
pointing
recent

when
understand ulte
meaning
is wrong AS
Grammatical level
is lost
grammar
almost useless

IMITATION

calculate
use algorithms
real numbers
$\sqrt{\,}$ etc…
picture
objects ⟹ letters
more abstract str ⟹ unknown
objects
which are elts
of a set itself
unknown
visual support
∃
anyone

Dem
properties
no one
taught them
unable
to produce
grammar
of prose texts
non their basics
- to reason properly
- verifying proofs
- control —
"therefore"
"because"
if O only if
iuui


FOUNDATIONS CRISIS 1
GEOM
(PYTHAGORE)
FLOOR A PRIORI KNOWLEDGE
in Aristotelian tradition
+ tradition of rationalism
NEWTON
INT CALC
(LAW GRAVITY) 1600
MATH / PHY — symbiosis
AFTER
NEXT 2 CENTURIES
question
seemed simple
perfect knowledge
= math of nature
have perfect a priori knowledge
among sci
able to mathematize
were most effective for predicting phenomena
could found a posteriori knowledge on a purely rational logic
This spirit
(CONDORCET)
Marie Jean-Antoine de Caritat
slowly the universe
Being solved one after the other a set
partial differential equations
1st BREAK
rotation of hyperbolic
IMPACT
the existence of mutually inconsistent axiomatic sys
each could be a model
→ R - math theoretical sys
Euclid // axiom
replaced
"consistent"
Riemann
Lobatchevsky
1/2 half of 19th
most on sphere

FOUNDATIONS CRISIS 2

Pre axiom period
proofs based on commonly accepted intuitions
(which could not be regarded as valid / axiom)

Responses and pb
rigour
trying to apply axiomatic not in all maths

Moreover
intuition into question

However

New way
spectacular success :)
∴ price
axiom consistent with phen exp
↓ dizzying contradictions with this new

EUCLIDE
only possible
one could believe that postulats (euc geom) a priori perfect on geom in real world
⚠ → 3 geom

why we choose
Axioms geom
plane
spherical
hyperbolic
as real assumptions
∀3 are consistant
we can not choose any a priori as foundation
Choice must be empirical
based on predictive power in a given situation

? PHYS
choose a formalism
But it was already accepted widely if not unconsciously that need to be was based on human ignorance

produce
one could infer the right choice from principles 1st
& logic or good math
a priori assumptions of reality
(to be confirmed afterwards

any sufficiently powerful axiom system to include integers numbers
had to be either inconsistent (contain contradictions)
or incomplete (too weak to decide the rightness or falsity of some statement of the system)

(Heinlein)

+ today

"formalist" zero content system

"platonic" old trash

process discovery

previous beliefs in metaphys

any axiom or formalism

metaphysical could still be saved

consistent keystone

(Gödel)

proved that

~~[illegible]~~ ledrvist

lost word

:(

FOUND (W) 3

abandon

rupture

pure

applied

(New) (Leib) (Fourier) (Gauss) also phenomenal sci

not early modern

hot head

~~[illegible]~~ bad addiction to platonic attitude

research ceased to meet phys & ~~[illegible]~~ learned to put themselves in their behind

small set rules detached of a byone

showed of maths ~~[illegible]~~ could be simple truth or some bound of set & axiomatic

(Russell Whit) (Frege) (Cantor)

for beloved platonists

20th

FOUND CRISIS 4

many other systems

kind of self

Such exact predictive function

not been found & not seem plausible

& climate behavior) chaotic interdep

⇓

exact predic'n impossible (impractical but in principle)

why only one good choice?

SP

predicts aim very new particles?

1

answer

There are ways to axiomatize maths that mutually exclude themselves

complex ⟨ practice / in principle

mod mod — R — prediction marmara

difficult applicability

MANY ATTEMPTS TO ADVANCE as a priori knowledge

Rotary

numerous paradoxes

unable to decide which axiom sys describes real maths

hope that standards axiomatizations are not inconsistent but just incomplete

However on part of empiricism i.e. as theoretical construction tool

GR

QS

perception indispensable

out of intuition

only meditating deeply on their formalism & extending them by cardinal even bizarre

Scanné avec CamScanner

is any math statement provable...

& Godel proved math'ly that A is NO!!

return Q whether a math Q correctly formulated (law) admits an A?

Super all sets impossible

set theory

self ref is the center of mus logical pb! destroys

Logic (undecidable prop")

as a collection defined by common ownership!

challenges notion of a set

PARADOXES 1

illustrations a few paradoxical sta

class of "undecidable propositions"

our ~~classes~~ of class of classes

those that ⊃ themselves reflexives

class of non empty sets

class of classes

EX

2 types

∅ non ref

Q? is the class of itself self ref or ~~ref~~ ?

if ref it ⊃ itself

~~ref~~ & thus in class of non ref classes that it ...

contradictory

Russell's 2 variants

does the set of all sets that do not contain themselves contain itself?

A is: if "yes" then "no"
if "no" then "yes"

can religion create entity more powerful than him

NO then he is NT all powerful

yes then he is not all powerful

Those who do not shave themselves are shaved by the barber but not those who shave themselves

So who shaves the barber

A also undecidable

PA R
2

there will always be Qs unanswered
⇑
every system
(living lang or math tool)
based itself is
nec'y incomplete!

incompleteness

$\vdash$

$\exists P : \neg Dem(P) \wedge \neg Dem(\neg P)$

given P
whatever the set of axioms,
there exist proposition that
we will never be able to prove
as true or false

$a_{r\,3th}$

a_M^V (Ru) pn

Lihay
2 types catalogues

Lien Pa

Wittgenstein

just a picture
of it

maths
to have the
logic
but not to
talk about
it

Hypo questionally from: 6 → ~~7~~ years until 11–12

strict p^n al I^n

~~##~~ expression: P ⇒ Q is a predicative implica^n (hence term "predicate")

There is no case in this ex where we can state P without Q – ex strict implica^n

"Syllogism"

EX –

– Consider p^n (antecedent) P: X is a man" * it implies the following p^n (consequent) Q: X is mortal"

Deductive procedure is to hold as true, provisionally, this first proposal that we name in logic a "predicate"

& to draw all the consequence & nec. that is to say Blanché

for its implica^ns

not only of a real observa^n identify a causal explanation

HYPOTHETICAL DEDUCTIVE REASONING

~~Process~~

DEF

HYP

DEDUCTION PROCESS

THEORETICAL PROCESS

EMPIRICAL OBS^NS PROCESS

confirm → provisional acceptance

doesn't confirm → reject

changes in hypothesis

PROPOSITIONAL CALCULUS

= ________ LOGIC"

type calculation < decisions / testing procedures

help ~~???~~ expression (proposition) is true

to determine when & especially if it is always true.

D (#3)

D1 an expression that is always true whatever the content [illegible] of the variables that compose it named "valid expression"

"tautology"

"law of propositional logic"

2 always false "contradiction" "antilogy"

3 sometimes T "contingent exp"

________ F

4 "assertion" we can say unambiguously whether it is T or F

5 "object language" to write logical exp^ns

6 meta — talk about in everyday language

R1 there are exp^ns that are actually not assertions

for ex "this statement is false" paradox student neither T or F

R2 consider the logical exp^n

A

if it is a tautology we frequently note it

$\models A$ & $A \models$ contradiction

R3 contrary to prove in a gen way that an assertion is T but not that it is F (give just one ex)

PROP (per premises)

D #4

"prop^n" is a statement that has meaning

→ we can say univocally whether this

stat is T or F "law of excluded middle"

EXs

E1 "I lie" not a p^n (premise)

E2

It's a solution that fails since it violates its own premise...

D #5

a p^n in binary log "principle of non contradiction"

never T&F at the same t

thus, a property on the set of propositions

E is a app P from E to the set of "truth values T,F" {T,F}

$$P : E \longmapsto \{T, F\}$$

"associates subset" when p^n only generate a portion E' of E and vice versa

EX:

In $E = \mathbb{N}$.

if $P(x)$ states "x is even"

then $P = \{0, 2, 4, \dots, 2k, \dots\}$

which is indeed only ~~an~~ associated subset of E but of same cardinal

D #6

Let P be a property of the set E.

A property Q on E is a "negation" of P

if and only if, for any x on E

iff
idem

$Q(x)$ is F if $P(x)$ is T

$Q(x)$ T F

condition "truth table" of values

P	Q
T	F
F	T

False / True | explicit values

binary

P	Q
1	0
0	1

P and Q always have opposite Truth values

denote this kind of ~~a~~ stat

Q is a negation of P

$$Q \iff \neg P$$

"negation connector"

R expressions must be WFE

by Def any variable is a

thus $\neg P$ is a WFE

if P, Q are WF formulas then $P \implies Q$ is a WFE

"I am ~~lying~~"

Connectors

D#7

Let P and Q two properties set defined on the same set E

$P \vee Q$ is a stat prop on E defined by

- $P \vee Q$ is T if at least P or Q are T

OR

≡

- ___ F otherwise "disjonction"

or inclusive

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

exclusive either this or that but not both $\underline{\vee}$ $\oplus$

?

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

XOR

every natural number is either even or odd but not both

is even ~~is~~ xor odd

$P \longleftrightarrow P$

$Q \longleftrightarrow \varphi$

$P \vee Q \longleftrightarrow P \cup Q$

√ associative on $[P \vee (Q \vee R)] = [(P \vee Q) \vee R]$

D#8 AND conjunction

- $P \wedge Q$ is T if both prop P, Q are true func syl
- $P \wedge Q$ is F otherwise

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

distributive

$$[P \vee (Q \wedge R)] = [(P \vee Q) \wedge (P \vee R)]$$

$\wedge \quad \vee \quad \wedge \quad \vee \quad \wedge$

Def "negation"

$$\neg T = F$$

$$\neg F = T$$

"logical complement"

using not P $\neg P$ $\bar{P}$

"De Morgan Laws"

$$\neg (P \wedge Q) \Leftrightarrow [(\neg P) \vee (\neg Q)]$$

$$\neg (P \vee Q) \Leftrightarrow [(\neg P) \wedge (\neg Q)]$$

"logical implication connector"

"conditional" $\Rightarrow$ $\supset$ $\rightarrow$

Let P, Q 2 properties given on E

$P \Rightarrow Q$ is a property on E defined by

P1 $P \Rightarrow Q$ is F if P is T and Q is F

P2 —— T otherwise

Fact that P logically implies Q means that Q is T for any assessments for which P is T

"if..then"

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

in other terms a F proposition implies that any conclusions will always be T

"informal fallacy" null H

if the prop is T the implication can be T only if the result is T

only for implication that are T

$\rightarrow$ $\Rightarrow$

EX "If you get your diplome,
I buy you a computer"

of all cases, only one corresponds
to a broken promise

"You have your degree" => "I buy you a computer"?

Exactly this
- if you have got graduate for sure,
I will buy you a computer,
- If you are not so, I would nothing

implication gives us that any false, we can deduce any proposal (last 2 lines)

R informal fallacy social

prove your premise (null H)

EX false → any proposal can be inferred

2+2=5 → you are the pope

1 suppose 2+2=5
2 substract 3 from each members
→ 1=2

3 By sym 2=1
4 the pope and I are 2
so 2=1 pope = I
→ I am the pope

back bone
any proof
evidence
deduction

$$P \Rightarrow Q \Leftrightarrow [(\neg Q) \Rightarrow (\neg P)]$$
$$\Leftrightarrow [(\neg P) \vee Q]$$
$$\neg(P \Rightarrow Q) \Leftrightarrow [P \wedge (\neg Q)]$$

"logical equivalence connector" or "biconditional on"

$$(P \Leftrightarrow Q) \Leftrightarrow [(P \Rightarrow Q) \wedge (Q \Rightarrow P)]$$
$$- [(P \wedge Q) \vee (\neg P \wedge \neg Q)]$$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$Q \Leftrightarrow P$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	F	T

$P \Leftrightarrow Q$ means (when it's T) that

P and Q always have the same T value

or "P and Q are equivalent"

This is T if P and Q have the same value, F otherwise

of course (it is a tautology)

nec & suf $\neg(P \Leftrightarrow Q) \Rightarrow (P \Rightarrow \neg Q)$

nec and suf for Q

Q is a nec and suf condition for P

$\leftrightarrow$ T that are T

$\Leftrightarrow$

Conc is that the conditions of the types "nec" "suf" "nec and suf"

only if   if   if and only if

There are

1 $P \Rightarrow Q$ reflects fact

nec cond^n for P or

in other words

P is T only if Q is T

is usable, when $P \Rightarrow Q$ is $= 1$

only if Q is also $= 1$

if P is true then Q is T

2 $P \Leftarrow Q$ $Q \Rightarrow P$

suf cond for P

formalization natural language statements

1 "prop^al variables" P, Q, R any simple proposals

if the same variable occurs multiple times, each time it symbolizes the same proposal

2 $\neg \wedge \vee \Leftrightarrow \Rightarrow$

3 Punctua^n () organize reasoning

Scanné avec CamScanner

Descrip^n — symbol usage

"negation" unary monadic $\neg$ $\neg P$

"it is not raining" $\neg P$

T if and only if P is F (in this case it is F that is raining)

$\neg\neg P$ equiv to P

"Conjunction" or "logical product" $\wedge$ $P \wedge Q$

"every man is mortal AND my car loses oil"

$P \wedge Q$ is T if and only if P is T and Q is T

"disjunction" "logical sum" $P \vee Q$

$P \vee Q$ is T if and only if P is T OR Q is T or both are T

can understand OR in to 2 ways — inclusively — either or — exclusively —

inc $P \vee Q$ T if P is T or Q is T both

exc — not both

XOR alternative

"implication" "inference" $P \Rightarrow Q$

"If ... then ..."

F is P is T and Q is F

if result is T, $P \Rightarrow Q$ T

antecedent F always T

Pure bottle

is F if and only if its antecedent is T and its consequent is F

"implication" "equivalence" $P \Leftrightarrow Q$

... if and only if ...

and ... is equivalent to ...

equiv of 2 p is T if they have the same T value

identity

definitions

P	Q	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$	$\neg P$
T	T	T	T	T	T	F
T	F	F	T	F	F	F
F	T	F	T	T	F	T
F	F	F	F	T	T	T

$\therefore$ before a logical consequence
therefore $\because$ because

$$(P \Rightarrow Q) \Leftrightarrow \neg(P \wedge \neg Q)$$
$$(\quad) \Leftrightarrow (\neg P \vee Q)$$
$$(\ \vee\ ) \Leftrightarrow (\neg P \Rightarrow Q)$$
$$(\ \wedge\ ) \Leftrightarrow \neg(P \Rightarrow \neg Q)$$

$\wedge\ \vee\ \Leftrightarrow$: be defined using common operators $\neg \Rightarrow$ through equi laws between them

de Morgan

sub $\leftrightarrow$ conj

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$
$$\wedge \qquad \vee$$
$$P \vee Q \qquad \neg(\neg P \wedge \neg Q)$$
$$\wedge \qquad \neg(\neg P \vee \neg Q)$$

Decision procedures

properties (prop^er variables)

2 ways to establish that a p is a law of propositional logic
We can either

(1 ~~the~~ use non automated procedure
(2 —— axiomatic and demonstrative —

Non axiomatic procedural decisions

simplest matric calculation
"met of truth tables"

Construction
tautology
Contradiction
Contingent

At least 4 ways to combine p^al ~~variables~~
Variables, brackets and connectors

1 Wellformed statement Non sense. Neither T nor F $(\vee P) \varphi$
2 Tautology stat always T $P \vee \neg \varphi$
3 Contradiction —— F $P \wedge \neg P$
4 Contingent stat — Sometimes T / F $P \vee \varphi$

- mechanical
axiomatized not entirely mechanical
- when facing a WFE or function of truth
1st determine how many and which C+
p^al var
- ~~To~~ examine various arguments that form
the expression
→ table 2^nd col

→ axiomatic procedural decisions

→ T
law
and lemma

2 axiomatic systems
2 spec rules named "inference rules"
(intuitive)

1 "Modus Ponens"

if we prove A and $A \Rightarrow B$ then we can deduce B

minor premise — major of the MP rule

ex From $x > y$

And $(x > y) \Rightarrow (y < x)$

we can deduce

$y < x$

2 "Substitution"

in a scheme of axioms replace a letter by any formula at and that all identical letters are replaced by identical formulas

ex axiomatic system Russell and Whitehead adopts $\neg, \vee$ primitive symbols

and define $\Rightarrow \wedge \Leftrightarrow$

$(A \Rightarrow B) \Leftrightarrow \neg A \vee B$

$(A \wedge B) \Leftrightarrow \neg(\neg A \vee \neg B)$

$(A \Leftrightarrow B) \Leftrightarrow (A \Rightarrow B) \Rightarrow (B \Rightarrow A)$

5 axioms + 2 rule inf

~~prim~~ symbols

A1 $(A \vee A) \Rightarrow A$

A2 $B \Rightarrow (A \vee B)$

A3 $(A \vee B) \Rightarrow (B \vee A)$

A4 $(A \vee (B \vee C)) \Rightarrow (B \vee (A \vee C))$

A5 $(B \Rightarrow C) \Rightarrow (A \vee C)$

not indep of each other

to justify that $\neg A \vee A$ has a sense

(b) Lukasiewicz

D #10

A finite sequence of formulas $B_1, B_2 \ldots B_m$ is named a "proof" from assumptions/hypothesis A_1, A_2, A_i if for each i

"Quantifiers"

predicates allows to formally handle stat "there exists an x such that [x has an American car]

or "for all x {if x is a ...., then x ... must

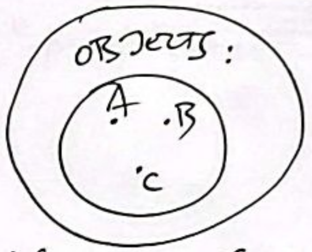
extend the composed formulas in order to assert existential quantifiers ("there...") — "for every)

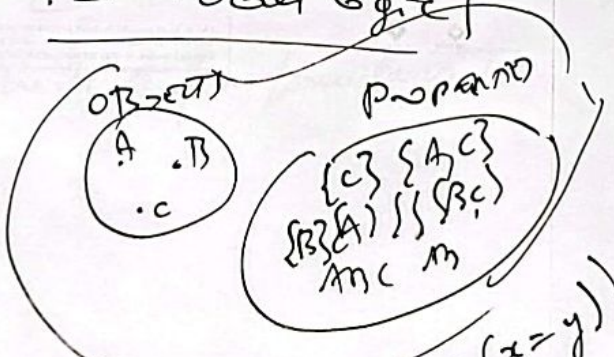
• predicate logic is named "1st order logic"

because it uses variables as basic mathematical objects — predicate FOPL

2nd → can also be sets

$\boxed{\text{First order logic}}$ $\boxed{\text{2nd order logic}}$





$\forall x : x = x$

$\forall x \forall y : ((\forall P : P(x) = P(y)) \to (x = y))$

D1 "universal quantifier" $\forall$ (for all)

D2 "existential —" $\exists$ (exists

ex

if any complex number is the product of a non negative number & a number of modulus 1 we will write

$\forall z \in \mathbb{C}, \exists r \in \mathbb{Z} \geq 0, \exists u \in \mathbb{C}: (|u| = 1 \; z = r u)$

order

ex $\forall x \in \mathbb{R}^+, \exists! y \in \mathbb{R}^*_+ : x = \ln(y)$

$\neq \exists x \in \mathbb{R} \; \forall y \in \mathbb{R} : (x \leq y)$

expression		is equivalent to
$\neg\neg A$	$\Leftrightarrow$	A
$\neg A \Rightarrow \neg B$	—	$B \Rightarrow A$

Predicate calculus

1st order logic proof & objects = formulas and proofs

terms and formulas are grammar

Grammar

D (#11)

D1 "terms" items want to prove some properties

- ely refers to elts of a set
- analysis real numbers "2nd order terms"

D2 "formulas" are properties of ~~objects~~ objects

- ely
- analysis exp continuity
- set theory $\subset$

D3 "proof" enable to check ~~that~~ if a formula is T

Language

D #12 a is the content of a family (not nec^y finite) of symbols

3 kinds of lang — symbols — terms — formulas —

R1 lang → "vocabulary" "or signature"

R2 "predicate" ← "relation" or calculus instead of 1st order logic

SYMBOLS # types

D1 "constant symbols" n neutral elt in set theory

D2 "~~fu~~ function symbols" or "functors" "ary" nb arguments

binary functor of multiplication × · in group G 2-ary binary operator

D3 "relation symbol" arity = binary.

D4 "individual variables"

∞ set V of variables

$x, y, z \quad x_1, x_2$

D5 $\neg \Rightarrow \vee \wedge \quad \forall \ \exists \ \exists!$

R1 ct symbol f^n symbol with 0 argument (arity zero)

R2 $\perp$ bottom atom FALSE

R3 Role function & relations very different

constant terms (of lang objects) — build formulas (properties)

Terms 1st order are the objects associated with lang

D1 ~~L~~ $\mathcal{L}$ lang set $\mathcal{T}$ of terms on $\mathcal{L}$

smallest set $\ni$ variables, cts & stable by applying function symbols of $\mathcal{L}$ to terms

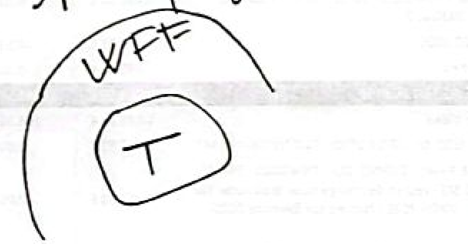
D2 "Closed term" only cts $\nexists$ var

D3 $\mathcal{T}_0 = \{t\}$ var or ct

for any $k \in \mathbb{N}$

$$\mathcal{T}_{k+1} = \mathcal{T}_k \cup \{ f(t_1, \dots t_n) \mid t_i \in \mathcal{T}_k \}$$

Formulas symbols & strings of symbols



Proofs

1) only finite set of rules
2) same rules for nodes & plugs

R

Rules of proofs